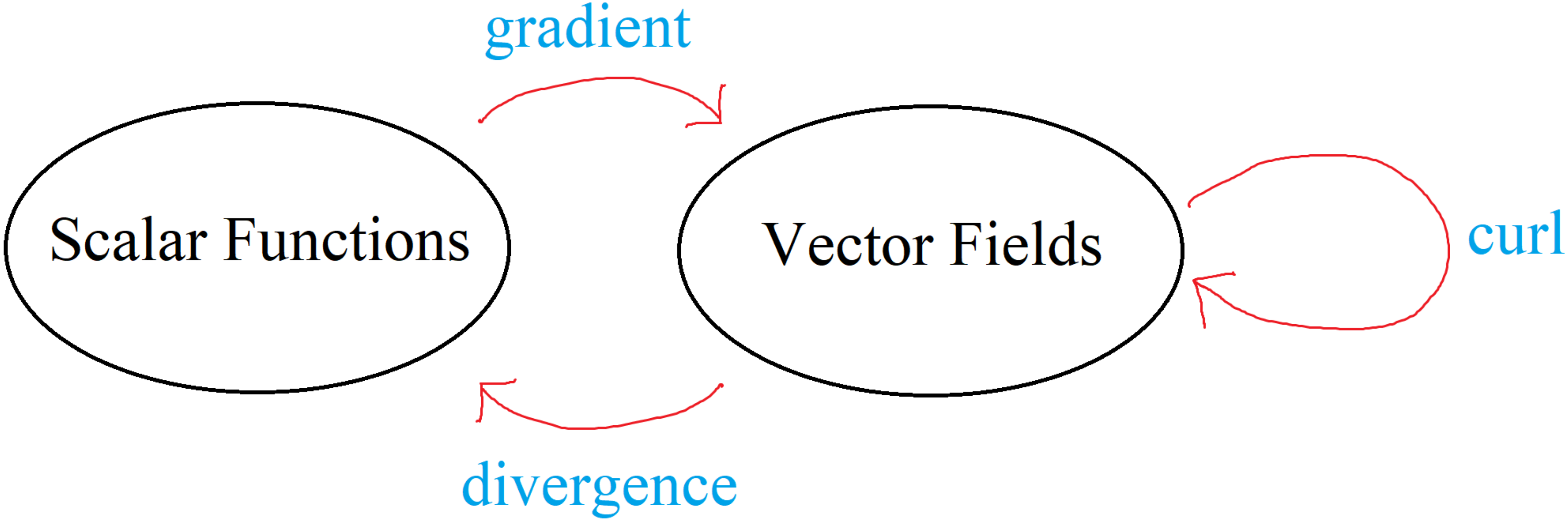
Section 16.5:
Curl and Divergence

What We’ll Learn In Section 16.5

1. The Del Operator
2. Gradient, Curl, and Divergence (general)
3. Curl / Results About Curl
4. Divergence / Results About Divergence
5. Updated Green’s Theorem Notation


gradient
Scalar Functions
Vector Fields
curl
divergence

1. The Del Operator

- An operator is a function whose inputs are functions and whose outputs are functions.
- The most common one is the derivative operator $\frac{d}{dx}$, but there are many others. (Examples on board)
- Operators don’t really make sense on their own. They are just objects waiting around for you to give them a function, then the operator will “act” on the function and transform it into another function.
- When an operator acts on a function, the notation we use will look like multiplication, but it’s not.
- Operators should always be written on the left of the function that it is acting on.

1. The Del Operator

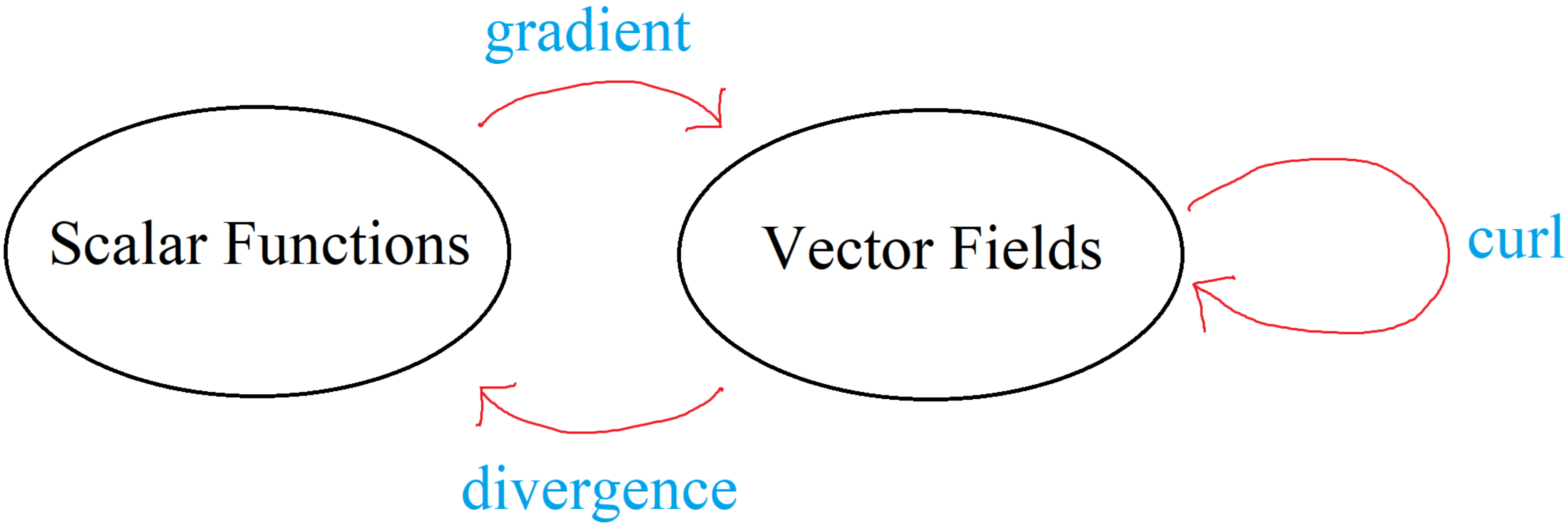
The <u>del operator</u> , notation ∇ , is the vector with the operators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$ in it in that order.

$$\nabla \equiv < \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} >$$

On it's own, it doesn't mean much. But… it makes sense when you combine it with a scalar (function) or another vector (field).

3 ways of multiplying with vectors…

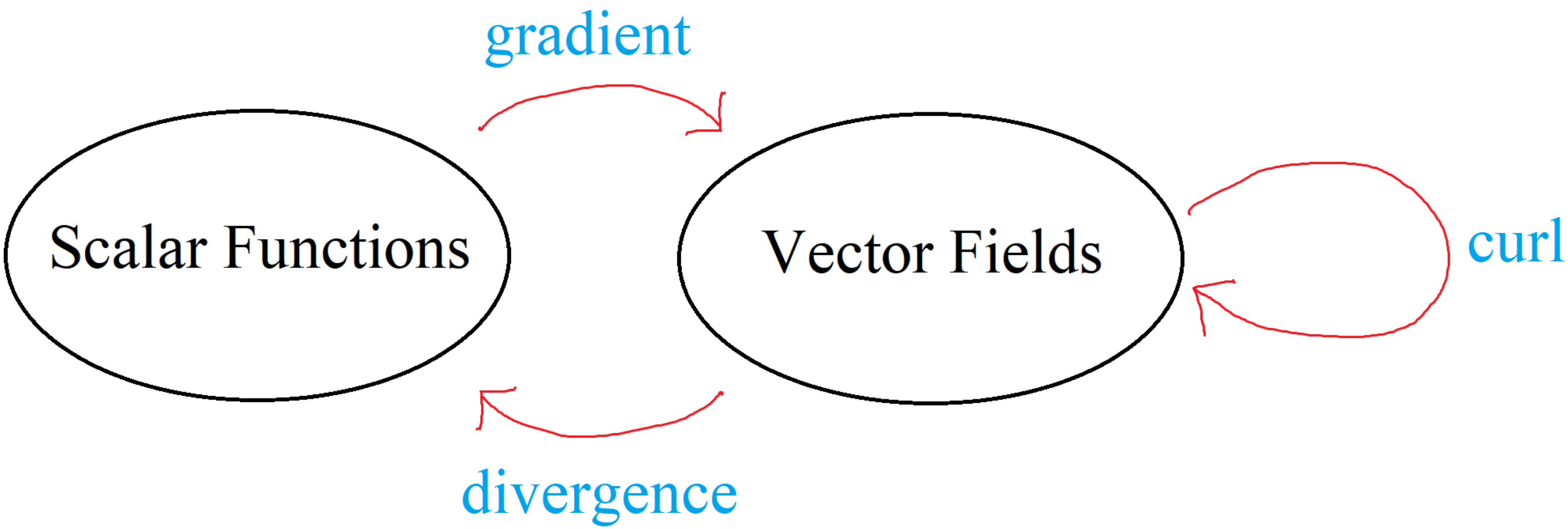
2. Gradient, Curl and Divergence (general)



<u>Gradient</u> $\quad \operatorname{grad} f = \nabla f$

$$= < \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} > f = < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >$$

2. Gradient, Curl and Divergence (general)

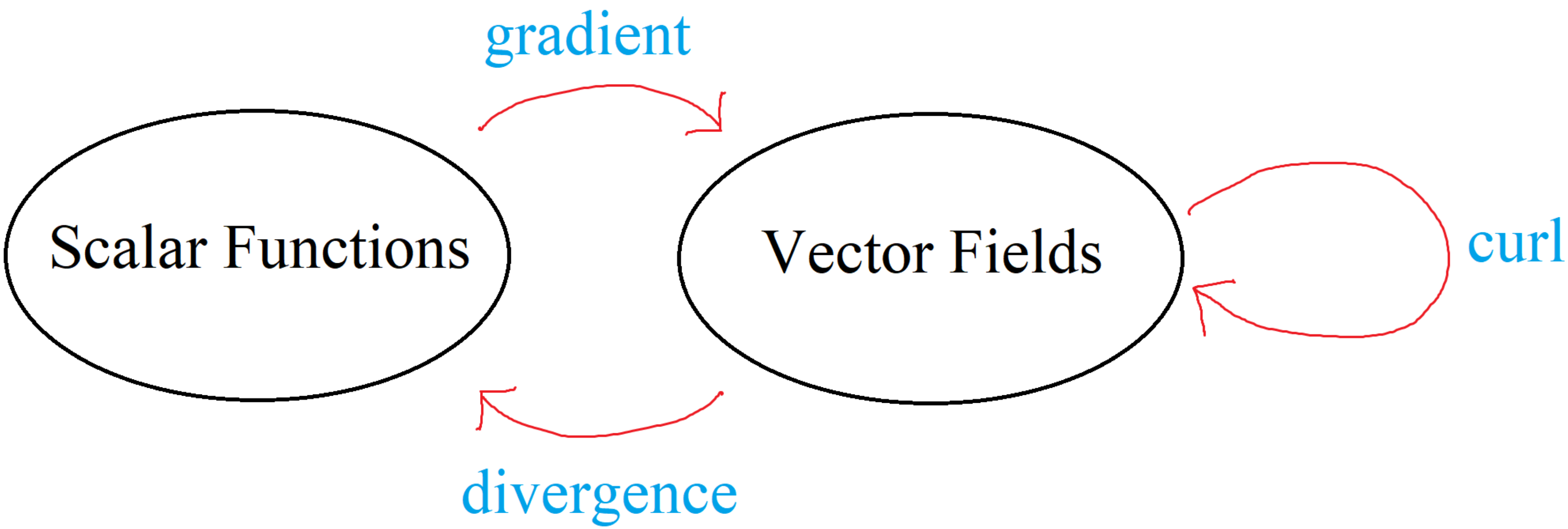


<u>Curl</u>

$$\text{curl } \vec{F} = \nabla \times \vec{F}$$

$$= < \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} > \times < P, Q, R >$$

2. Gradient, Curl and Divergence (general)



<u>Divergence</u> $\qquad \operatorname{div} \vec{F} = \nabla \cdot \vec{F}$

$$= \left< \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right> \cdot \left< P, Q, R \right> = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

3. Curl / Results About Curl

Ex 1: If $\vec{F} = < xz,\ xyz,\ -y^2 >$, find curl $\vec{F}$.

3. Curl / Results About Curl

3 Theorem

If f is a function of three variables that has continuous second-order partial derivatives, then

$$\boxed{\operatorname{curl}(\nabla f) = \mathbf{0}}$$

Proof ?

Note: This gives us a way to show that a 3-component vector field is NOT conservative.

3. Curl / Results About Curl

Ex 2: Show that the vector field $\vec{F} =< xz,\ xyz,\ -y^2 >$ is not conservative.

3. Curl / Results About Curl

Recall: Results from section 16.2…

5 Theorem

If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

6 Theorem

Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

3. Curl / Results About Curl

For 3-component vector fields, we now have…

3 Theorem

If f is a function of three variables that has continuous second-order partial derivatives, then

$$\boxed{\operatorname{curl}(\nabla f) = \mathbf{0}}$$

4 Theorem

If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\operatorname{curl} \mathbf{F} = 0$, then $\mathbf{F}$ is a conservative vector field.

3. Curl / Results About Curl

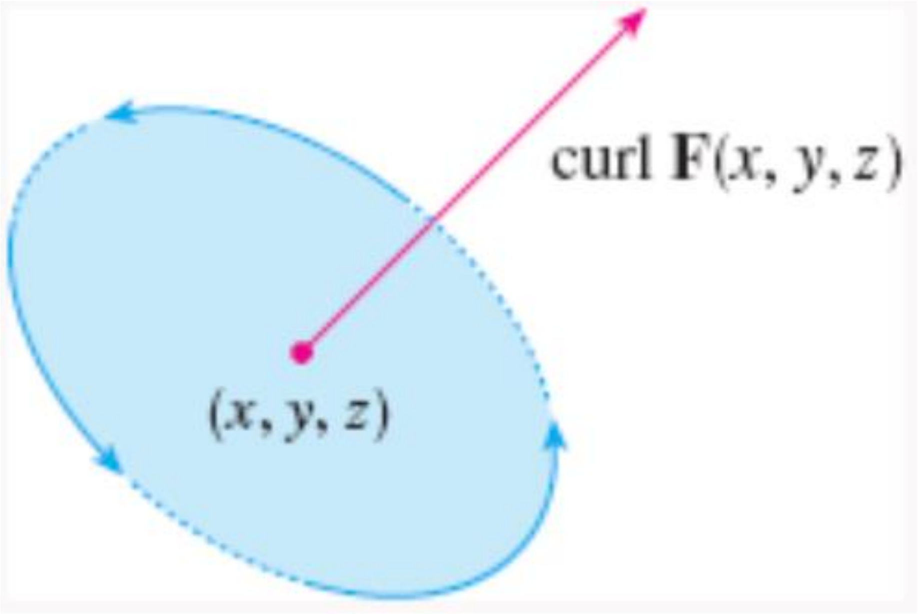
Ex 3:

a) Show that $\vec{F} =< y^2z^3,\ 2xyz^3,\ 3xy^2z^2 >$ is a conservative vector field.

b) Find a function f such that $\vec{F} = \nabla f$.

3. Curl / Results About Curl

Why the name Curl?



4. Divergence / Results About Divergence

Ex 4: If $\vec{F} =< xz,\ xyz,\ -y^2 >$, find div $\vec{F}$.

4. Divergence / Results About Divergence

11 Theorem

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\boxed{\operatorname{div}\operatorname{curl}\mathbf{F} = 0}$$

Proof ?

4. Divergence / Results About Divergence

Ex 5: Show that the vector field $\vec{F} =< xz,\ xyz,\ -y^2 >$ can't be written as the curl of another vector field, that is, $\vec{F} \neq curl\ \vec{G}$.

5. Updated Green's Theorem Notation

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\,dA$$

$$\oint_C \mathbf{F} \cdot \mathbf{n}\,ds = \iint_D \operatorname{div} \mathbf{F}\,(x, y)\,dA$$

5. Updated Green's Theorem Notation

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA$$